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ABSTRACT

The purpose of this paper is to investigate how explanation is interactively constituted in two mathematics instructional sequences during one year-long teaching experiment in a second grade classroom. The position taken is that classroom norms for what constitutes an acceptable explanation are a joint construction between teacher and students. Previous work shows that in inquiry-oriented mathematics classrooms, students shift from using social reasons to justify mathematical activity to giving explanations based on experientially-real mathematical objects. Explanation itself becomes an object of reflection during this process. The focus on explanation in this study is extended to include an analysis of the relationship between children's explanations and the classroom mathematical practices that emerge as the instructional sequences progress. Findings significantly demonstrate the emerging nature of explanation and clarify the interrelationship between explanation and other classroom mathematical activity. Contains 23 references. (Author/DDR)

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Explanation as an Interactive Accomplishment: A Case Study of One Second-Grade Mathematics Classroom

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The purpose of this paper is to investigate how explanation is interactively constituted in two mathematics instructional sequences during one year-long teaching experiment in a second-grade classroom. The position taken in this paper is that classroom norms for what constitutes an acceptable mathematical explanation are interactively constituted by the teacher and the students. These norms, which are specific to the mathematical aspects of the children's activity, are called sociomathematical norms to distinguish them from the social norms that regulate classroom activity more generally (Yackel & Cobb, 1996). Previously, we have shown that in inquiry mathematics classrooms students shift from using social reasons to justify their mathematical activity to giving explanations based on experientially-real mathematical objects. In the process, explanation itself becomes an object of reflection for students. For example, as the school year progresses, students initiate discussions about the adequacy of the explanations of their peers (Yackel, 1992).

In this paper, I extend our work on explanation to include an analysis of the relationship between children's explanations and the classroom mathematical practices that emerge as the instructional sequences progress. The significance of this research is that it demonstrates the emerging nature of explanation and clarifies the interrelationship between explanation and other classroom mathematical activity. Further, it highlights the importance of attending to the sociomathematical norms that are constituted in classrooms.

Method and Data

The Teaching Experiments

The data for this paper are taken from two mathematics teaching experiments conducted in a second-grade classroom in an inner-city school in the Midwest. Each teaching experiment was from three to six weeks in duration. The first teaching experiment was conducted in the first three months of the school year and the second in the last two months of the school year. An additional teaching experiment, conducted in the same classroom in the middle of the school year, is not discussed in this paper. The purpose of

each of the teaching experiments was to investigate the children's mathematical learning during a specific instructional sequence and, at the same time, to develop and refine the instructional sequence through a process of developmental research (Gravemeijer, 1994). The pragmatic purposes of each of the teaching experiments may be viewed as assisting the teacher to develop an inquiry form of instructional practice (Cobb, Wood, Yackel, & McNeal, 1992) through use of innovative instructional activities, and assisting the students to develop sophisticated understandings of number and numerical relationships. Each of these purposes was largely achieved over the course of the school year.

The classroom teacher provided the instruction for most of the lessons. On occasion, one of the researchers would intervene in the instruction in some way. Typically, these interventions were in response to requests for assistance from the teacher. The intervention might be to interject one or more comments or questions in whole class discussion or to lead the class discussion for a brief period of time. During the second teaching experiment, the teacher was preoccupied with a school committee assignment that took her away from her classroom frequently and made it difficult for us to reflect and plan with the teacher. As a result, on several occasions, one of the researchers actually conducted a segment of the lesson in its entirely while the teacher observed. During these lesson segments, we attempted to capitalize on opportunities to notate students' solution methods and, by doing so, inducted the teacher into the use of unconventional but pedagogically-productive notational methods. The teacher commented on the benefits of such observation. These opportunities to observe her students engage in mathematical thinking and activity without having to assume responsibility for leading the class were invaluable to her reflection. Despite these interventions, it is accurate to describe the classroom mathematics instruction as provided primarily by the classroom teacher. The teacher had taught second grade for a number of years and had participated with us in mathematics teaching experiments in previous years. She and the researchers had developed a close working relationship. This relationship accounts, in part, for her

openness in asking questions of the researchers during a lesson. For example, on one occasion when the teacher had not anticipated the solution methods students had given and was perplexed by how to proceed next, she simply turned to the researchers and said, "What do I do now?"

There were approximately 20 students in the class throughout the school year. The student population was relatively stable. Only one or two students left the class and three new students joined the class. All of the students and the teacher were from the same minority group and the same cultural community. Thus, sensitivity to cultural diversity was not an issue in this study.

Data

All mathematics lessons conducted during the teaching experiments were video-recorded for later analysis. Video-recordings include the small group work of selected pairs of students as well as all whole class activities and discussions. In addition, individual interviews were conducted with the students both before and after the teaching experiments. The data corpus consists of sets of videotaped recordings, field notes, copies of student work from all of the lessons and interviews, and transcripts of selected classroom episodes.

Instructional Sequences

The instructional sequences for the two teaching experiments were derived, in large part, from several instructional sequences first developed at the Freudenthal Institute. In this case, we were attempting to use the teaching experiments to engage in developmental research (Gravemeijer, 1994) so that we might study students' learning while, at the same time, revising and refining the instructional activities for the local school setting.

Fall teaching experiment. The instructional sequence for the first teaching experiment was made up of two components. The first component was *Patterning and Partitioning* and the second was *Structuring Numbers*. The instructional intent of the *Patterning and Partitioning* component was that students develop number concepts based

on grouping, as opposed to counting, strategies (Gravemeijer, Cobb, Bowers, & Whitenack, in press). Activities included finger patterns, dot patterns, ten-frames, and scenarios that engender imagery of splitting a quantity into two quantities. For example, in one scenario, students were asked to imagine a group of people on a double-decker bus (van den Brink, 1989), and to figure out various possibilities for how many people might be on the upper deck and how many on the lower deck of the bus.

The patterning activities first grew out of our earlier work in this and another school in the same community. We became aware that a number of students entering second grade did not have visual imagery to support their early number concepts. As a result, their primary mode of operating was to count. We were interested in fostering concepts of number based on number relationships and grouping strategies so that children's modes of operating would include other possibilities in addition to counting. As a result, we designed a number of activities that use dot patterns and other types of visual displays of numbers organized around 5 and 10 to help children develop imagery for number concepts (Yackel, E., & DiNuzzo, N, 1991). In a subsequent teaching experiment conducted by Cobb, Whitenack, and McClain at another site, finger patterns were used as the basis for developing imagery for number concepts (Whitenack, 1995).

The instructional intent of the second component, *Structuring Numbers*, was that students develop flexible methods for operating with numbers under 20, based on five-, ten- and doubles-referenced strategies (Gravemeijer, 1991). Many of the instructional activities in this component involved use of a device called an arithmetic rack.[1] Our own previous experience with the arithmetic rack indicated that the instructional activities had the potential to foster children's development of thinking strategies (Yackel, 1995).

Spring Teaching Experiment. The instructional sequence for the teaching experiment conducted during the last two months of the school year was called the *Empty Number Line*. This sequence was first developed at the Freudenthal Institute by Treffers and De Moor (1990) and follows the Realistic Mathematics Education (RME) instructional

design theory. The instructional intent of this sequence is that students develop flexible ways of adding and subtracting numbers up to one hundred, including conceptualizing differences as situations for filling up the gap as well as situations for taking away. For example, to solve a problem such as, "Mike has to travel 82 miles to get to his cottage. He has gone 57 miles. How many miles does he yet have to travel to reach the cottage?", a student might figure out the gap between 57 and 82 as opposed to taking 57 away from 82.

This instructional sequence makes use of a beadstring that consists of 100 beads composed of ten groups of ten beads, with groups alternating in color. For example, the beadstring might have 10 black beads, followed by 10 white beads, followed by another set of 10 black beads, etc. In the initial stages, students use the beadstring to solve tasks. As the instructional sequence progresses, students use a number line with graduated markings in lieu of the beadstring, and later still use an "empty number line" to solve tasks. The empty number line differs from the traditional number line in that it does not contain graduated markings. Instead, it is used as a way to record solution methods that might be thought of as moving forward (adding on), moving backward (taking away), or filling the gap (finding the difference). See Figure 1. Finally, the empty number line serves to orient

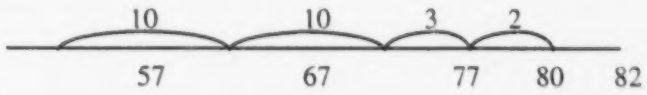

Figure 1. An empty number line solution to finding the gap between 57 and 82.

students' thinking and becomes a model for their mathematical reasoning. Thus, there is a gradual progression from using the beadstring, to using a graduated number line, to using an empty number line, and finally to working independently of the number line, though possibly using it to orient solution processes.

The empty number line teaching experiment did not follow the instructional sequence as intended for a variety of reasons. Limited use was made of the beadstring.

Instead, the empty number line was developed in close conjunction with an instructional activity, called Target, that had been used from the beginning of the school year. In this activity a target number is identified and students are asked to figure out how much needs to be added to a given number to reach the target. For example, the target might be 40. The task might be to figure out what needs to be added to 28 to reach the target of 40. In this activity, the teacher typically poses many different tasks using the same target, frequently sequencing the tasks to create opportunities for thinking strategies. For example, with a target of 40, the sequence of numbers selected might be 28, 18, 8, 9, 19, 29. The Target activity was used in the project classroom throughout the year. However, as the year progressed, the instructional intent of the activity changed. Early in the school year, it was used to foster development of partitions of small numbers and of thinking strategies. Later, it was used to foster development of compensation strategies for units of ten. In the last two months of the year, it was used as a means of linking the empty number line symbolism and notation to children's solution methods. In particular, the empty number line provides imagistic support for sequence-based, as opposed to collections-based, solutions.[2]

In summary, there were significant departures from the intended empty number line instructional sequence as it was realized in the classroom. Some of these changes were necessitated by pragmatic issues rather than by pedagogical considerations. Nevertheless, the sequence was beneficial for many of the students who developed flexible sequence-based solution methods for two-digit addition and subtraction problems, including difference problems.

Framework for analyzing Classrooms

The complexity of classroom research presents a challenge for the researcher not only for data analysis, but also for conceptualizing the analysis process itself. Here we follow the interpretive framework developed by Cobb and Yackel (1996). This framework coordinates a psychological perspective with a sociological perspective at the level of the

classroom. A key aspect of our framework is that we posit a reflexive relationship between the sociological constructs and their psychological correlates. To clarify, we consider the relationship between the sociomathematical norms that are constituted in the classroom, a sociological construct, and the individual participants' mathematical beliefs and values, the psychological correlate. To say that these two sets of constructs are reflexively related means even more than that they are mutually enabling and constraining. It means that one literally does not exist without the other. For example, as children give explanations that they deem viable, they are not only participating in the taken-as-shared normative understandings of what constitutes acceptability, they are also contributing to the ongoing negotiation of what is taken as normative. Thus, their (individual) mathematical beliefs and values develop concomitantly with the classroom sociomathematical norms. In the same way, children's individual mathematical conceptions and activity, a psychological construct, and the classroom mathematical practices, the sociological correlate, are seen as mutually constitutive.

By classroom mathematical practices we mean those aspects of the classroom mathematical activity that are normative. For example, in one second grade classroom that we have studied, as the year progressed, it became normative to decompose two-digit numbers into tens and ones, operate on the tens and ones, and recombine the individual results to get the final result. For example, 38+47 might be solved by decomposing the 38 into 30 and 8, the 47 into 40 and 7, combining the 30 and 40 to get 70, the 8 and 7 to get 15 and adding 70 and 15 to get 85. As such activity became normative, it was beyond justification. That is, students no longer felt obliged to explain such details as that 38 can be decomposed into 30 and 8. It was taken-as-shared that numbers can be decomposed in this way. We would say that decomposition of numbers into tens and ones had become a taken-as-shared classroom mathematical practice.

While we take the relationship between classroom mathematical practices and individual students' mathematical conceptions and activity to be reflexively related, the

particular relationship between any specific classroom mathematical practice and specific mathematical conceptions and activities is open to empirical verification. That is, if teachers have particular classroom mathematical practices as a goal and work toward the negotiation of those in the classroom, we might ask what specific mathematical conceptions students develop. Conversely, if students develop certain mathematical conceptions, we might ask what types of classroom mathematical practices become normative. Thus, we might conjecture that students develop collections-based conceptions of number as they participate in the classroom mathematical practice of decomposing numbers into tens and ones and recombining them in flexible ways (Yackel, Cobb, & Wood, in press). In fact, the empty number line sequence has as its instructional intent to foster students' development of flexible sequenced-based conceptions of number to complement collections-based conceptions of number that they developed as they participated in a prior instructional sequence called the candy shop (Cobb et al., 1991).

Mathematical Explanation and Justification

Mathematical explanation and justification can be studied from a number of different perspectives. In this paper, our interest is in mathematical explanation and justification as an interactional accomplishment and not as logical argument. Our focus is on what the participants take as acceptable, individually and collectively, and not on whether an argument might be considered valid from a mathematical point of view. Krummheuer's work on argumentation (1995) and Yackel and Cobb's work on sociomathematical norms (1996) provide a background for the approach we take in this paper. Each is discussed briefly here.

In his study of the ethnology of argumentation, Krummheuer analyzes argumentation using Toulmin's scheme of conclusion, data, warrant, and backing. According to this scheme, the conclusion is a statement that is made as though it is certain. Krummheuer, following Toulmin (1969), refers to the support one might give for the conclusion as data. If one were to say, "P therefore Q," P represents the data and Q

represents the conclusion. Warrant refers to the rationale that might be given to explain why the data are considered to provide support for the conclusion. Krummheuer uses the example of two boys who are working collaboratively to solve the problem 4 x 4 = _____. The boys decide the answer is 16 because 8+ 8 = 16. According to Krummheuer's scheme, this information, 8 + 8 = 16, is the data. As Krummheuer points out, it is legitimate to ask why it is that 8 + 8 has anything to do with the product 4 x 4. The rationale that is given is the warrant for the data. The two boys give us some insight into what serves as a warrant for them when they say , "It's 4 sets of fours ... 'Cause 4 sets, um 4, 2 sets make 8." In these statements, the boys explicate why 8+8 is relevant to the given problem. In doing so, they provide a warrant. Backing provides further support for the warrant, that is, the backing indicates why the warrant should be accepted as having authority (Toulmin, 1969). In this case one of the boys holds up his fingers as he says, "You have 2 more sets. Like it's 2 and 2 make 4." Krummheuer interprets this action as indicating that for these boys, 2 and 2 make 4, provides further explanation of why it is appropriate to figure out the value of two sets of four and two more sets of four. Thus, in this example, this action on the part of one of the boys forms a backing for the warrant.

In summary, Krummheuer uses the notions of data, conclusions, warrants, and backing as a means to analyze argumentation and to explicate how argumentation is an interactive constitution of the participants. For him, an argumentation in any given situation "contains several statements that are related to each other in a specific way and that by this take over certain functions for their interactional effectiveness" (Krummheuer, 1995, p. 247). Statements do not have a function apart from the interaction in which they are situated. Thus, what constitutes data, warrants, and backing is not predetermined but is negotiated by the participants as they interact.

Krummheuer's approach to argumentation is useful here for two reasons. First, it provides a way to demonstrate the changes that take place over time. In particular, we will demonstrate that what constitutes warrant and backing for students changes over time.

Second, this approach to argumentation helps to clarify the relationship between the individual and the collective, in this case between the explanations and justifications that individual children give in specific instances and the classroom mathematical practices that become taken-as-shared. As mathematical practices become taken-as-shared in the classroom, they are beyond justification and, hence, what is required as warrant and backing evolve. Similarly, the types of rationales that are given as data, warrants and backing for explanations and justifications contribute to the development of what is taken-as-shared by the classroom community, that is to the mathematical practices of the classroom.

Yackel and Cobb (1996) refer to normative understanding of what constitutes an acceptable mathematical explanation and justification as a sociomathematical norm and argue that such normative understanding is negotiated as the teacher and students interact in the classroom. For example, in the inquiry classrooms they have studied, including the classroom that is the subject of this paper, it became normative that explanations and justifications increasingly involved describing actions on what we could describe as experientially real mathematical objects for the students. Consequently, explanations that did not carry the significance of actions on taken-as-shared mathematical objects that were experientially real for the students were frequently challenged. (These challenges, in turn, gave rise to situations for the teacher and students to negotiate what was acceptable as a mathematical explanation.) This sociomathematical norm is critical for the discussion in this paper because it helps to clarify how individual children's explanations and the classroom mathematical practices are interactively constituted.

The negotiation of sociomathematical norms that occurs in classrooms takes place largely through the medium of discourse. It is as the teacher and the students talk about their mathematical thinking and activity that normative understandings emerge. In this sense, our perspective on how sociomathematical norms are constituted is consistent with dialogical perspectives on discourse elaborated by Nystrand (1992) and Rommetveit

(1992). In this perspective, discourse is seen as a progressive modification and expansion of a temporarily shared social reality. Rather than a predetermined cultural practice that individuals learn to participate in (Nystrand, 1992), it is a dynamic and evolving enterprise. Accordingly, we could have chosen to discuss the constitution of the sociomathematical norm of what constitutes an acceptable explanation and justification in terms of the evolving and emerging discourse in the project classroom.

Results and Discussion

Mathematics instruction in the classroom that is the subject of this paper followed an inquiry form of instruction (Cobb et al., 1992). The teacher had used this form of instruction in previous years, but all of the children in the class had experienced only traditional mathematics instruction previously. Consequently, the teacher had to initiate the negotiation of classroom social norms and sociomathematical norms characteristic of inquiry instruction with an uninitiated group of students. Expectations of developing personally meaningful solutions, explaining one's thinking, listening and attempting to make sense of others' explanations, and asking questions when you do not understand or want clarification, all needed to be established. Similarly, norms for what constitutes an acceptable mathematical explanation and justification needed to be negotiated. While such norms are continually negotiated and renegotiated in any classroom (Yackel & Cobb, 1996), the initiation of certain normative understandings require special effort to counter the beliefs that students have developed in traditional instruction. For example, in traditional instruction students are seldom expected to explain, especially if they give a correct answer, and explanations that are given typically are recitations of solution procedures (Cobb et al., 1992). In inquiry classrooms, mathematical explanations are expected to be descriptions of actions on experientially real mathematical objects. Thus, an initial step in negotiating the sociomathematical norm of what constitutes an acceptable mathematical explanation involves establishing that explanations should have a mathematical basis.

Initiating the Negotiation of What Constitutes Mathematical Explanation

In the project classroom this negotiation began the first day of the school year. Several of the activities used at the beginning of the school year involved dot patterns. In one activity, dot patterns were flashed on an overhead screen several times, each time for only a few seconds. The children's task was to figure out how many dots there were in the pattern and to describe (explain) how they "saw it." The instructional intent of the task was to foster development of visual imagery and to generate discussions about number relationships for small numbers. For example, one of the first patterns used on the first day is shown in Figure 2. In response to the question, "How did you see it?", several

Figure 2. Dot pattern for three shown on the overhead screen.

children gave replies such as, "I looked at them," "With my eyes," "I saw three going in a slant," "I saw a slanted line," and "I counted." These replies formed the basis for initiating the constitution of the task as a mathematical one, meaning that, in this case, the descriptions (explanations) should be numerical. They should be about quantity. Thus, the response "I saw three going in a slant" was acceptable but "I saw a slanted line" was not. Similarly, a response such as "I counted them" was acceptable but "I looked at them" was not. This apparently trivial example demonstrates that what is considered mathematical communication in the classroom must be constituted. As students participate in the classroom activity and discussions, they learn what constitutes mathematics in this classroom. At the same time, their activity contributes to what is constituted as acceptable mathematical activity in their classroom. In this example, students both learned and contributed to what constituted an acceptable mathematical description (explanation).

The next day, the teacher displayed the dot pattern shown in Figure 3. This time,

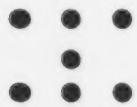

Figure 3. Dot pattern for seven shown on overhead screen.

children's responses included both counting and non-counting explanations. For example, several students reported that they saw seven *because* they saw it as three and one and another three. Interestingly, none of the children who gave such a response explained how he/she arrived at a total of seven based on the three, one, and three. These children seemed to take it for granted that their explanation was sufficient. That is, they took it for granted that the other students would immediately know that 3 + 1 + 3 = 7. Yet we know from individual interviews conducted with the children at the beginning of the school year and from their mathematical activity in the classroom that some of the students in the class would have to count by ones to solve this addition problem. Nevertheless, all of the children in the class were able to readily see that there were "three dots in the first row, one in the second row, and three in the third row" (at least one child referred explicitly to the number of dots in each row of the pattern in this way).

Thus, we might assume that in Krummheuer's scheme, the statement that 3 +1 + 3 = 7 would serve the function of data for the conclusion that there were 7 dots in the pattern. No warrant would be necessary. A warrant would be needed if children had no way to understand why the sum of 3 + 1 + 3 was relevant to answering the question of how many dots there were. Thus, following this assumption, we might presume that all of the children had a way of making sense of the relevance of the explanation. This would be significant because it would mean that even those children who would have to count by ones to solve 3 + 1 + 3 could understand that solving this addition problem was a viable alternative to counting the dots by ones.

On the other hand, these responses had the potential to contribute to children's interpretation of the task as having two components: reporting information about the arrangements of the dots and figuring out the totality of dots. However, the instructional intent of the task was that students would be able to use the visual imagery to think about numbers in various ways, including thinking of a quantity as composed of other smaller quantities. It was not that they engage in the two separate tasks, reporting an arrangement and reporting the totality. Thus, the expectation was that the configuration of dots a child would report would indicate something about how the child figured out the totality. Expectations such as these have to be established. For example, in this and future lessons some children reported how they viewed the arrangement of dots without relating it in any way to the total.

A case in point is Nina's solution given later in the lesson on the second day when the teacher flashed the dot pattern shown in Figure 4. Nina reported only that she saw 4 at the top, 1 in the middle, and 4 at the bottom. Her response implies that she assumed that this was an adequate response. From this information we are unable to determine if Nina knew that there were 9 dots in all and if so, how she figured it out. She may have been

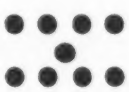

Figure 4. Overhead dot pattern for nine.

simply reporting how she "saw" the pattern. Furthermore, since her response was accepted by the teacher with no further probing, we do not know if, for Nina, there was a relationship between the four, one, and four that she reported and the totality of dots. Thus, we cannot assume that for Nina four and one and four served as data for the conclusion that there are 9 dots. Furthermore, we cannot assume that the statement, there are nine dots *because* $4 + 1 + 4 = 9$, would have explanatory relevance for Nina. For her, a warrant may be needed. Thus, the earlier explanation that there are 7 dots (in Figure 3)

because there are 3 dots, one dot, and three more, may not have been adequate for Nina. In this example, we have pointed to a critical feature of explanation. That is, the adequacy of an explanation depends on all of the participants in the interaction, not only on the person giving the explanation. In this sense, we are treating explanation as a communal, not an individual, activity (cf. Cobb et al., 1992).

Several of the responses that children offered for the task posed next in the second lesson were particularly useful to clarify the interrelationship between the arrangement of dots reported and ways of figuring out the totality. In this case the dot pattern shown was for the number ten. It was identical to the pattern for nine except that it had two dots in the middle row. The first child to respond to the dot pattern for ten reported that she counted them by ones. The next two responses were given by Anita and by Denzel.

A: I knew that 4 plus 4 is 8 and I counted 2 more and that made 10.

D: Put 2 from the middle at the top. That makes 6 and the other 4 makes 10.

In these responses, both Anita and Denzel explain how they used the configuration as they "saw" it to figure out how many dots there were in the pattern. That is, they provide the explanatory relevance of the configurations they use, 4, 4, and 2 for Anita, and 4, 2 and 4 for Denzel. In fact, we might infer that their interpretations of the pattern were based, in part, on number combinations that they were able to use to determine the sum. As Labinowitz (1985) points out, what we "see" is based on what we understand, not the other way around.

In this instance the teacher did not capitalize on these responses. The students were left to make sense of these explanations for themselves. Voigt (1995) notes that such incidents serve an important function in supporting students' mathematical learning by making it possible for them to become aware of more conceptually advanced forms of mathematical activity while, at the same time, leaving it to them to decide whether to take up the intellectual challenge. In this case, some of the children would still need to count the dots by ones to determine the total. However, we would argue that, because these

explanations include explicit information about how the configurations were used in determining the sum, even these children might make sense of the relevance of these more sophisticated solution methods and might begin to reconceptualize their own counting activity in terms of the quantities in the individual rows of the configuration.

Despite the fact that the teacher did not call attention to these solutions in this instance,[3] such explanations became more commonplace. Within a few weeks it became normative for students to give explanations of that type in both the dot pattern activity and other visual imagery activities, such as those that involved single and double tens frames. We would say that describing visual patterns in terms of smaller quantities and giving explanations of how to find the total using those quantities had become a taken-as-shared classroom mathematical practice.[4] In Krummheuer's terms, we might say the explanatory relevance of statements such as "There are 7 dots because there's three in the top row, one in the middle, and three in the bottom row" was apparent to an increasing number of students. A warrant was not needed to support this as data for the conclusion that there are seven dots.

Thus far, we have taken a close look at one classroom mathematical practice and how it emerged in the classroom. The explanations that were given by some of the children in the class were central to the constitution of the practice. In turn, as the classroom practice became taken-as-shared, normative understandings of what constitutes an acceptable explanation in this situation also evolved. In the remainder of the paper we look across each of the two teaching experiments to see what practices developed and how these relate to the children's evolving understanding of explanation.

Children's Evolving Explanations

Fall Teaching Experiment. As the *Patterning and Partitioning* and the *Structuring Numbers* instructional sequence progressed, other classroom mathematical practices were constituted in the classroom. Across the teaching experiment, these practices also emerged: describing solutions by relating problems to previously solved problems; describing

configurations (such as double tens frames configurations and arithmetic rack configurations) in terms of fives, tens, and doubles; using the imagery of the arithmetic rack or of double tens frames to explain number relationships and thinking strategies; and acting with known number relations to solve problems. In each case, the constitution of the practice was facilitated by explanations given by individual children. To use Krummheuer's terms, we would say that the evolution of classroom mathematical practices parallels (is enabled and constrained by) an evolution of what children take as data, warrants and backing in argumentation. This, in turn, is closely related to the individual children's conceptual understandings and possibilities. In the following discussion, we outline the evolution of the practices that emerged in the fall teaching experiment and relate these to the nature of the children's explanations.

One practice that emerged was describing solutions by relating problems to other previously solved problems. Some children's use of thinking strategy explanations for double tens frame tasks and for the Target activity, described earlier in this paper, contributed significantly to the negotiation of this classroom mathematical practice. By the end of the second week of the school year, a number of the children were participating in the aforementioned practice of describing visual patterns in terms of smaller quantities and giving explanations of how to find the total using those quantities. The teacher then introduced the double tens frame to provide opportunities for children to relate visual patterns to fives, tens, and doubles. One instructional activity that the teacher used was to place some chips in each frame of a double tens frame and to flash the image briefly, several times, on the overhead screen. The task was to figure out how many chips there were and to explain how you figured it out. The teacher used chips of two colors, one color for the left frame and a different color for the right frame. In doing so, she created a situation where children might interpret the task as finding the sum of two quantities—the quantity in the left frame and the quantity in the right frame. An important feature of the instructional activity was that the tasks were sequenced so that the configurations in

subsequent tasks might be thought of as related to those in prior tasks. We would say that the teacher created opportunities for students to relate tasks to each other, without requiring them to do so. For example, in the ninth lesson of the school year the teacher posed six tasks in the following two sequences:

First Sequence

Task 1: five chips in the left frame and five in the right frame

Task 2: six chips in the left frame and five in the right frame

Task 3: six chips in the left frame and four in the right frame

Second Sequence

Task 4: nine chips in the left frame and three in the right frame

Task 5: eight chips in the left frame and three in the right frame

Task 6: eight chips in the left frame and four in the right frame

Several children gave responses in which they related tasks. For example, when called upon to explain his solution to Task 2, Hakeem replied that "Five and five is ten and ... it was one more dot on the six. One more dot on the five and that made 6 and I put them together and that made 11." Later, when questioned by another student, he elaborated, "Because watch. I got 10 and then she added one more and look, then it went to the 6 and this was 5 and then this goes to this (demonstrating with his hands, first holding up five fingers and then changing to six) and that made, that made 11." This elaboration was particularly informative in the class discussion since Hakeem had previously explained his solution to Task 1 by holding up both hands and remarking that "then I looked at all of them. That's how I know it was ten." Thus, his two explanations taken together could be interpreted by others as clarifying both that he saw the tasks as related and also how he used the relationship to solve the second task. Similarly, he solved Task 3 by relating it to Task 2. Another student solved Task 4 by relating it to Task 3. Such direct references to previous problems became important in the class discussion as other students took note and subsequently attempted to use a similar strategy. In this lesson, one boy specifically

commented on Hakeem's solution to Task 2 as insightful, describing it as a "genius" solution.

It is useful to note that several students also used solution methods which capitalized on the structure of the tens frame. For example, two children solved Task 4 by mentally moving one chip from the right frame to the left frame to fill up the ten and conceptualized the problem as ten and two more. Thus, solutions that, from our perspective, are thinking strategy solutions began to emerge in the double tens frame overhead task. However, at this stage these solutions may have been dependent on the availability of the visual material.

Over the next few weeks, increasingly many children began to give solutions of this type. In addition, the way they explained their thinking became more cryptic. For example, in contrast to Hakeem's explanation in which he explicitly said that there was "one more *dot on the five and that made 6,*" explanations (for other problems) such as, "It's just one more," "You had 11 and all you did was add one more," "One more. So 16," became more commonplace. These slight differences may seem insignificant. Yet, following Krummheuer's analysis of argumentation, we infer that increasingly these cryptic statements served as data for the conclusion without the need of a warrant or of additional backing.[5] This evolution in what constituted sufficiency for the children is precisely what makes it possible for us to refer to the classroom mathematical practices that emerged as taken-as-shared. We also note that while these explanations are based on tasks that are posed with visual material, these cryptic explanations do not refer explicitly to the visual material. Increasingly, the explanations were about number and number relations.

We note that it might also be argued that, for at least some of the children, these apparently increasingly sophisticated explanations might only be explaining the relationship between problems and might not signify a totality that is obtained from the two quantities in the individual tens frames. The data indicate, however, that for most tasks some reference was made during the discussion to the individual quantities in the two tens frames mainly

because there were children who were not yet able to participate in the classroom mathematical practice that was emerging. These children were still counting either the visual material displayed on the screen or the visual material in conjunction with fingers which they used as substitutes when the overhead projector was turned off. These references to the individual quantities contributed to the ongoing interpretation of the task as figuring out the total of the two quantities shown, even when the method of solution was to relate the task to a prior task.

The teacher also began using the instructional activity called Target during this time. In the course of this activity, children began relating problems by noticing patterns in the sequences the teacher presented and by using the compensation strategy. This instructional activity also contributed to the classroom mathematical practice of describing solutions by relating problems to previously solved problems. Since the types of explanations children engaged in are similar to those described above we will not elaborate on this point.

We conclude this section by discussing the types of explanations children developed when using the arithmetic rack and the emergence of the classroom mathematical practice of using the imagery of the rack (or of double tens frames) to explain number relationships and thinking strategies. In the initial arithmetic rack activity, children worked individually with their racks to show a given number of beads in any way they wished, and then participated in a whole-class discussion of how they thought about which beads to move. The types of solutions children gave ranged from relatively sophisticated methods based on five- and ten-referenced strategies to moving the beads individually while counting by ones. In addition, a number of children used thinking strategies in their explanations. The extend of such use made it clear that children were capitalizing on the taken-as-shared understandings of number and number relationships that had begun to emerge from their prior experiences with the double tens frame and Target instructional activities. Since the task was to create a number on the rack and describe how you thought about it, the explanations children gave typically made explicit reference to the beads on the

rack. Nevertheless, they differed qualitatively from each other. In addition, since the teacher repeatedly asked if anyone had shown the number in a different way, some of the children may have reconceptualized their initial activity so that they could report it as different. This had the effect of raising the level of the discussion (Cobb, Boufi, McClain, & Whitenack, in press) to one in which students were asked to reflect on their prior activity and compare it to that of others. The net effect was that several of the children reported relatively sophisticated strategies. These solutions demonstrate what these children took as shared and also indicate how the classroom mathematical practice of using the imagery of the rack to explain number relationships and thinking strategies developed.

In the following dialogue, three different children give their solutions to the task of showing 12 on the arithmetic rack. In each case the teacher was the only one to comment on the solutions. The full text is included here.

T: Louis, how did you figure yours out? Tell us how you figured it out without counting.

L: Cause I just knew that you said that it was, uh, 5 black ones, uh, 5, uh, 5 dark ones and 5 light ones and I knew to put all the brown ones to the side and put one light one with, uh, the 5 and that made it 6 and I did the same thing at the bottom.

T: Okay. So he said that he knew that all five of-- these, err, dark ones was 10 and he said he didn't have to count because he just pushed all of this over (moving 5 dark beads to the left on each of the upper and lower rods on her overhead arithmetic rack) and that would be 6 and one here and that would be 6 (moving one light bead to the left on each rod) and he would have 12. He did that without counting. Did anybody else do that without counting?

From Louis's explanation we are able to infer that he thought about the problem as 6 and 6 even though his description is limited to explaining how he could figure out how to move six beads without counting them individually by ones.[6] Interestingly, he did not go on to explain why 6 and 6 is relevant. He seemed to take it for granted that the other students would know, as he did, that 6+6=12. By this time, a number of children did

know that 6+6=12. However, for those children who did not know that 6 and 6 is 12, Louis's remarks might seem irrelevant. The teacher's repetition did not provide direct verbal clarification of the relevance of 6 and 6, for she simply restated that there would be 6 and 6 and that would be 12. However, her actions on the rack may have been useful because she pushed all of the dark beads over at once while saying "...these dark ones was 10" and then moved one light bead on each rod while saying "and that would be 6 and one here would be 6." Thus, some of the children may have been able to figure out that 10 and one and one is 12 and then look back at the configuration to "see" the six beads on each of the rods. In this sense, the teacher's actions in conjunction with her remarks may have served the function of providing a warrant and additional backing for the warrant. That is, her actions and remarks might have explained the relevance of 6 and 6. For other students, the teacher's actions and remarks might have led them to reflect on and develop a variety of number relationships.

The next child to respond to the task was Anita.

A: I did, uh, put 8 at the top and 4 at the bottom and I knew that was 12 because you can put 2 more and then, and then you put the bottom, and that's 12, too.

T: So how did you know that you had 8? How did you figure your 8 out? Did you count that?

A: No.

T: Tell us how you figured it out.

A: I could put the 2 at the top and so you, but I put 4 at the bottom and 8 at the top.

T: Okay, so you had 5 brown ones and 3, 5 dark ones and 3 light ones and then you put what at the bottom?

A: Four.

T: And then you put 4 at the bottom. Without counting she knew that. Hakeem?

Our interpretation of Anita's first remark, based on other solutions that she had given in class previously, is that she was attempting to explain that she might move two

beads from the bottom to the top to get 10 on the top and 2 on the bottom. However, she never clarified this explicitly. We would infer that few children, if any, were able to interpret Anita's explanation in terms of a filling-up-the tens strategy. The teacher provides no additional support for Anita's explanation. Instead, the teacher asks her to clarify how she knew she had eight without counting. This task was easy for Anita, who simply referred to the colors of the beads as "5 brown ones and 3 white ones." Later in the same lesson Anita gave a similar explanation for how she showed the number 13. This time she was more articulate in her explanation.

A: I put 8 at the top and 5 at the bottom.

T: And how did you figure that out? Eight at the top and 5 at the bottom.

A: Because I knew if I would have added 2 more I would make 10 and I put 3 at the bottom and that was 13.

The initial episode continues as Hakeem explains his solution method. From his description we infer that he showed 12, using 7 beads on the top rod and 5 on the bottom rod.

H: Ah, I know, I know, I had 5 brown ones and then I know if I put 2 more then that would equal 7 and then I know I took one off the 7 in my brain then I put on the --, I had 5 down here and then I put it on down here.

T: Okay, so you had 5 dark ones and 2 light ones which was 7 and then you put 5 down here.

H: Yes.

Hakeem's explanation, like Anita's, might be seen as extremely terse. From the observer's perspective, we would say that Anita reconceptualized her solution(s) using a going-though-tens strategy, while Hakeem used a compensation strategy, relating 7 and 5 to 6 and 6. We know from our observations in the classroom that by this time in the year he and many of the other students in the class "just knew" that 6+6=12. However, his cryptic and somewhat inarticulate explanation left much to the listener. In this case, no one,

including the teacher, challenged Hakeem or demanded further clarification. It is likely that Hakeem's remarks did not serve as an adequate explanation for a number of students in the class. Had he said "I know that 6+6=12", his explanation could be interpreted as giving explanatory relevance for this statement. Without this statement, his comments might have been interpreted by other students as simply a remark about alternative ways to move the beads.

In the preceding discussion we have characterized each of the three children's explanations as rather cryptic. In addition, we have pointed out how these explanations might have been inadequate for some of the children in the class. On the other hand, these explanations also give us some indication of what the children assumed they could take for granted. For example, both Louis and Hakeem took it for granted that others in the class knew that 6+6=12. Anita seemed to take it for granted that others would know that 10+2=12. Both Hakeem and Anita assumed knowledge of the compensation thinking strategy. The fact that these explanations were not challenged contributed to the emergence of the classroom mathematical practice of using imagery, in this case, the imagery of the rack, to explain number relationships and thinking strategies. In fact, over the course of the next two to three weeks, it became normative to explain number relations and thinking strategies using the imagery of the rack and later to act with known number relations to solve problems, without reference to imagery. Thus, we would say that these became taken-as-shared classroom mathematical practices in this classroom.

Spring Teaching Experiment. The spring teaching experiment was designed around the *Empty Number Line* instructional sequence as previously described. The empty number line was used initially as a means of recording and notating children's solution methods during the Target instructional activity. As this method of recording and notating became institutionalized, the function of the empty number line changed. In addition to serving as a means of recording and notating students' solutions, the empty number line increasingly served the function of orienting the students' thinking as they went about

developing their solutions. The imagery of "hopping" forward and backward on the empty number line was clearly apparent in children's descriptions. This imagery, in turn, led to more sophisticated solution methods and ways of thinking. Thus, in Gravemeijer's language (in press) the empty number line changed in function from being a *model of* children's thinking to a *model for* more formal arithmetical reasoning. By the end of the teaching experiment, incrementing and decrementing with various number combinations in efficient ways to solve addition, subtraction, and difference problems was emerging as a taken-as-shared classroom mathematics practice. I say the practice was emerging rather than that it had emerged because it was only within the last two days of school that we have sufficient evidence to say that this way of acting was becoming normative for problems in general. Much earlier it had become normative to describe solutions in terms of the imagery of the empty number line in the situation-specific case of the Target activity.

Our experience with the empty number line illustrates how notating with non-conventional symbolization for the purposes of communication within the classroom can be a powerful pedagogical tool. As Gravemeijer et al. (in preparation) note, by negotiating a symbolization scheme, the teacher and students become aware of which aspects of their solutions need to be symbolized, and in doing so make specific aspects of their solutions explicit. Further, as students develop their own interpretations of the symbolizations of others' solutions, they might begin to use them in their own way to develop their own solutions and to explain them to their peers. Thus, we concur with Van Oers (in press) that use of the empty number line supports the development of flexible addition and subtraction activity. On one hand, the explanations students give can be interpreted as their attempts to develop meaning for the symbolization of the empty number line. On the other hand, these interpretations are means the students can use to develop solutions to other problems. In the remainder of this section, we describe the evolution of students' explanations over the instructional sequence. First, we describe how the empty number line was introduced in

the Target activity to notate students' solutions and explanations, and then we illustrate the types of explanations that were becoming common by the end of the school year.

Our use of the empty number line to notate students' solutions in the Target activity was prompted by our success with doing so in another setting. Thus, we expected that in this second grade classroom the empty number line would quickly become a means for thinking and developing solution methods. The following episode is taken from the first class period in which this approach was used. The target number is 72. The task is to figure out what number to add to 50 to get 72. In this case one of the researchers was the teacher. Figure 5 shows the diagram this researcher drew while redescribing Cecil's solution.

R: Do you have it figured out, Cecil?

C: You got to add 22.

R: Tell us how you figured that out. Cecil says we have to add 22. Tell us how you figured that out, Cecil.

C: Because, because if you have 20, if you have 50 plus 20 that will be 70 and if you add 2 more that will be 72. That's how I did it.

R: Let's see if I can kind of show what Cecil said. Cecil said that if we are at 50 and if we add on 20 we get up to 70. (The researcher drew the number line and the arc joining 50 and 70 and labeled it 20.) Is that what you said, Cecil?

C: Yes.

R: And then to get, if we add on two more we get to 72. (The researcher drew the portion of the diagram showing the 2.)

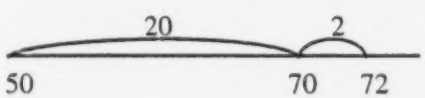

Figure 5. The researcher's notation for Cecil's solution to a Target task.

R: So Cecil said if we add 20 and two more that's like adding 22, so he said 50 and 22 is 72. How many of you agree with Cecil?

In saying that she was going to "show" what Cecil said, the researcher introduced the diagram as a means of notating his method. By asking him to confirm that her repetition was what he said, she further linked the diagram with his method. Finally, by saying, "So Cecil said if we add 20 and two more that's like adding 22, so he said 50 and 22 is 72," she was attempting to relate the written symbols 20 and 2 on her diagram to his answer of 22. In a sense, the researcher was telling the students how to interpret the diagram as a record of Cecil's solution.

The empty number line was used in the same way to notate other children's solutions throughout the Target activity in this and in future lessons. In each case, the researcher was careful to use notation that was consistent with the solution each child gave. For example, in a subsequent task in which the problem was to figure out how much to add to 48 to get to the target of 72, Louis and Max gave two different solutions. Their explanations and the surrounding dialogue are included below. The empty number line notation the researcher drew for these two solutions is shown in Figure 6. In each case, the notation parallels the way the child reported thinking about his solution.

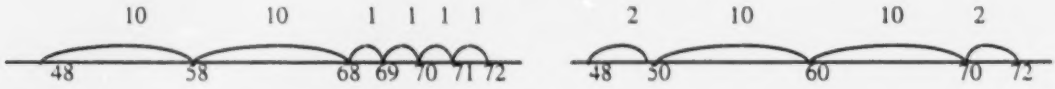

Diagram for Louis's solution. Diagram for Max's solution.

Figure 6. Two empty number line diagrams.

Louis: I said it's 24. How I got 24 was I said 48, 58, 68, 79, 79 (touching a finger with each count).

R: (interrupting) 69. Do that over. Say it again.

L: I said, 48, 58, 68, 69, 70, 71, 72.

R: Let's see if I can write this. You said 48, 58, how many is that? (The researcher draws the arc from 48 to 58.)

L: Ten. (The researcher writes 10 above the arc.)

R: And then you went to 68, and how many was that? (The researcher draws the arc from 58 to 68.)

L: Twenty.

R: That was 20 in all and then you went by ones, didn't you?

L: Yeah.

R: You went 69, 70, 71, 72. I can't squeeze that in here but, so you went a ten and another ten and four ones and that's how you got 24. (The researcher attempts to draw in the four arcs of length one unit each.)

After an intervening solution by another student that did not lend itself to notating on the empty number line, Max offered that he had another way to solve the problem. The researcher drew the diagram, notating Max's solution piece-wise as the discussion progressed, in the same way as she had done in the episode with Louis..

M: I have another way.

R: You have another way?

M: (Nods his head, yes.) Cause you got 48 and then you add on, you add on two more, you get a two more and then a ten, you add a ten.

R: Max said, add on two more and then let's add on ten. When we add two more on, what will we be up to? Who can help us out? Forty-eight plus two more, what will we be up to?

M: Twenty-two.

R: What would you be up to, Shaquille?

S: Fifty-two, I mean 50.

R: We'd be up to 50. Okay. And then we add on ten more, what would we be up to?

S: Sixty.

R: Sixty. Max, I'm sorry I interrupted you. I hope you remember what you were going to say. What was the rest of what you wanted to say?

M: And then you go, you add another one ten and then you get up to 70.

R: Another ten takes us up to 70.

M: And then you add on a two.

R: And then you add on two. That was a little bit different than Louis's but it was still the same total wasn't it. Two tens and four ones for twenty-four.

Up to this point, we have focused on the way the empty number line was introduced in the class discussion. We turn now to consider the nature of the explanations the children gave. In the case of Louis's explanation, he took it for granted that other children knew that the difference between 48 and 58 is 10 and that the difference between 58 and 68 is 10. This was the case for approximately half of the students in the class. However, for others, this was not true. Some of the latter children came to understand this only as the empty number line instructional sequences progressed. Further, Louis took it for granted that he did not have to explain that 10+10+1+1+1+1=24. The researcher's elaboration of Louis's solution, with his assistance, served several purposes. First, it linked the solution to the empty number line notation. Second, it made explicit those aspects that Louis took for granted. Further, these two aspects of the elaboration were intertwined. For example, by asking Louis, "You said 48, 58. How many is that?" and then writing 10 above the arc on the diagram after Louis responded, "Ten," the researcher was helping to clarify the relevance of "48, 58, 68, 69, 70, 71, 72" to Louis's conclusion that the answer is 24, and at the same time to relate the notation to Louis's verbal response. In the same way, the elaboration (interruption) of Max's solution served the function of making explicit that Max's method made use of the decade points, 50, 60, 70.[7] The researcher did not give Max the opportunity to explain if or how he figured out the total distance traversed but, instead, provided that clarification herself when she explained that it was the same total as Louis's.

Over the next several days, the symbolization of the empty number line as a means of notating solutions for Target problems continued to be negotiated by the researcher and the students. For example, explanations that simply reported the endpoints of a series of hops that ended at the Target but that did not report the total distance traversed were challenged. Within a few days, students began to develop efficient methods such as the following solution given by Charelle to the problem of figuring out how much to add to 41 to reach the target of 60.

C: 19 because 51, and I added on 10 more and I took 1 away.

The researcher drew the diagram shown in Figure 7 to notate Charelle's solution. In

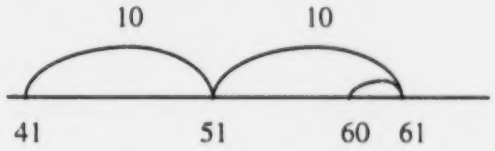

Figure 7. Researcher's empty number line notation for Charelle's solution.

this way the researcher introduced a notational means of recording both the action of adding ten and the action of taking away one.[8] In this case, as in the first example discussed, the diagram was introduced naturally as a means of notating the student's solution. It should be noted that throughout the Target instructional activity students solved problems using only mental arithmetic. No paper or pencil were available. Thus, the students had to interpret the notation used by the teacher or researcher but did not use it themselves. However, they had opportunities to use it themselves on small group and individual tasks.

We do not know if the solution Charelle gave in the above example was prompted by the empty number line notation, especially since notation that indicated both the actions of adding and subtracting on the same problem had not been used in class before. However, by the end of the year, the explanations children gave indicated that they made efficient use of the "hopping" back and forth notions, as illustrated by the following

examples taken from the second to last day of the school year when the problems were posed as bare number sentences.

To solve the problem 46+19, Charelle gave the following explanation.

C: I said 46+10. That was 56 and I added on 9 more.

R: Charelle, how did you know what that gave you for an answer? What was your way of figuring it out without counting by ones?

C: Because I said, another big 10 and then I went back one.

The teacher had posed the problem 46+19 on the chalkboard and asked students to figure out how to solve the problem using the empty number line. However, they were expected to work mentally. They had no paper and pencil to work with. Thus, the solutions they developed involved anticipating the result of using the empty number line or working with mental images of it. They did not have the opportunity to actually carry out various steps piece-wise and then use the results in subsequent steps. Thus, Charelle's comment, "I said another big 10 and then I went back one," indicates that she was able to use the imagery of the empty number line to add 9. Anita gave a similar solution to the same problem.

A: I said I added 19. I took a big hop of 19.

R: Explain how you figured out that it was 65.

A: Because I said if I would have added 20 it would have been 66 but all I added was 19.

R: Okay. So that's a nice way. A way to show somebody else how you thought about it is to say you took a big jump of 20 and you backed up one. (The researcher drew an empty number line diagram showing a forward jump of 20 from 46 to 66, marked it +20, and a backwards jump of 1, from 66 to 65, and marked it -1.)

A: I just went 19.

This exchange between the researcher and Anita shows that she was not dependent on the imagery of the empty number line to add 19. When the researcher attempted to recast her explanation in terms of the empty number line, "you took a big jump of 20 and you backed up one", Anita countered, "I just went 19." Thus, for her it was no longer

necessary to explain the relationship, 20-1=19. Further, she took it for granted that others, like she, had curtailed compensating actions on the number line. At least three other children in the class made use of such compensating actions in their explanations during this lesson. However, most of them were explicit in describing them. For example, to explain how they solved 53-9, Charelle and Shaquille said, "We said 43 and one more is 44." Thus, we would say that these ways of acting were emerging as a classroom mathematics practice. Other ways of acting that were becoming commonplace involved incrementing and decrementing by ten or multiples of ten within decades and partitioning numbers in efficient ways to capitalize on decade points. Consider Hakeem and Don's solution to the same problem 53-9. Hakeem reported, "First I, -- 53 and then I went back 3, I went to 50 and then I'm going back 6 more ... at 44."

We note that in these explanations, what the children took for granted was very different from what they took for granted at the beginning of the school year. Charelle and Shaquille took it for granted that they did not have to explain that 9 is 10-1. Further, they did not even refer to taking away ten. They only referred to the result, 43. They assumed that in doing so, others would know that they had taken away 10. Similarly, Hakeem took it for granted that he did not need to explain the relevance of the 3 and the 6 to his solution. He not only took it as shared that 3+6=9, but also took the relevance of this partition of 9, as opposed to other possible partitions, as apparent. That is, he took it for granted that he did not need to explain the rationale for getting to a decade point. Further, he assumed that it was not necessary to explain how he figured out that 53-3=50 or that 50-6=44.

The examples we have given were selected to illustrate the way the empty number line was introduced as a means of recording and notating students' solutions and how students used the imagery of acting on the line in their mathematical reasoning and explanations. By the end of the instructional sequence, incrementing and decrementing with various number combinations in efficient ways to solve addition, subtraction, and difference problems was emerging as a taken-as-shared classroom mathematical practice.

The changes in the explanations the children gave, in particular, in what they assumed they no longer had to clarify or make explicit, indicate the way in which the development of explanation and the constitution of classroom mathematical practices are mutually supporting and constraining.

Concluding Remarks

In this paper we have investigated how explanation was interactively constituted in one mathematics instructional sequence at the beginning of the second grade school year, and the relationship between children's explanations and the classroom mathematical practices that emerged in this and another instructional sequence. We have demonstrated that what children take for granted evolved as each sequence progressed and across sequences, from the first to the last. Classroom mathematical practices themselves emerge as what is taken for granted evolves. This is significant because it helps us develop accounts for how it is that children learn as they participate in the communal activity of the classroom. Our use of taken-as-shared classroom mathematical practices as a construct highlights the fact that not all of the students in any given class share the same mathematical understandings. The notion of taken-as-sharedness is brought into focus precisely by those situations in which a member of the community, typically a student in classroom situations, acts in ways that are inconsistent with the practices that have become normative. In such cases, the contrast between these inconsistent actions and normative actions serves to make the features of the normative practices explicit. In this way, these inconsistent or contrasting actions contribute to the constitution of the normative practices. We did not include such cases in this paper but such cases are available in the data. For every classroom mathematical practice that we mentioned, there were numerous instances where children were acting in ways inconsistent with the practice. Thus, we do not claim that we have outlined a learning path that all of the students in the class followed. All we claim is that as students' explanations evolved, certain types of mathematical activity became normative in the class.

As a final comment, we note that our discussion points to the critical role played by the explanations of individual children in setting the stage for and helping to initiate the negotiation of normative mathematical activity. In this sense we have demonstrated the importance of having in the class one or more children present whose mathematical ways of acting and thinking are insightful. The solution methods and explanations these children give can provide the basis for initiating taken-as-shared classroom mathematical practices.

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End Notes

[1] An arithmetic rack consists of two identical rods of ten beads each, a group of five dark beads and a group of five light beads. Initially, all of the beads are positioned at the right of the rack. To show a quantity, that number of beads is moved to the left of the rack. For example, to show the quantity 7, one might move 5 beads on the upper rod and 2 beads on the lower rod to the left side of the rack.

[2] By sequence-based solutions, we mean those that take an initial quantity without decomposition as the basis for the solution. For example, if we solve 23 + 58 by beginning with the 23 and adding 50 to get 83, then adding 8 more to get 81, we call the solution sequence-based. If we add 20 and 50 to get 70, 3 and 8 to get 11, and then combine the 70 and 11 to get 81, we say the solution is collections-based.

[3] In our interactions with the teacher, we encouraged her to call attention to solutions such as this as a means of advancing the mathematical agenda. At times, we intervened in the class when we felt that an important opportunity might otherwise be missed.

[4] To say that something is a taken-as-shared classroom mathematical practice does not mean that all members of the class participate in the practice in the same way. Some children may be unable to participate because their conceptual understandings are not as advanced as those of others. For example, in the project classroom some children were still counting by ones well into the school year. These children did not participate in the classroom mathematical practice described here. It is such discrepancies that make the notion of taken-as-shared practices meaningful. For example, when children attempt to participate but do so in inappropriate ways, other children challenge them. These challenges help to clarify that the practice has become normative. There are expectations that the group, as a collective, has for how to participate in the practice.

[5] Hakeem's explicit reference to one more dot on the five provides a rationale for why it is that 5 and 5 is ten is relevant. For some children, this additional information would be needed as a warrant.

[6] In fact our data do not include a record of how Louis actually moved the beads. He may have moved them one by one or he may have moved each group of five as a unit. He may even have moved all six at once. In any event to report his solution he had to (re)conceptualize the six as a composite unit of 5 and one more.

[7] The researcher was aware that some of the children could count by tens only on the decade numbers and therefore saw Max's solution method as one that might be viable for those students, while Louis's solution method would not be. In fact, later in the lesson the researcher called special attention to Max's solution, labeling it as "Max's method," and asked other children to think about how they would solve another problem using Max's method.

[8] Later, a negative sign preceding a number was introduced to signify the action of taking away.